

Survival probability and first-passage-time statistics of a Wiener process driven by an exponential time-dependent drift

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The survival probability and the first-passage-time statistics are important quantities in different fields. The Wiener process is the simplest stochastic process with continuous variables, and important results can be explicitly found from it. The presence of a constant drift does not modify its simplicity; however, when the process has a time-dependent component the analysis becomes difficult. In this work we analyze the statistical properties of the Wiener process with an absorbing boundary, under the effect of an exponential time-dependent drift. Based on the backward Fokker-Planck formalism we set the time-inhomogeneous equation and conditions that rule the diffusion of the corresponding survival probability. We propose as the solution an expansion series in terms of the intensity of the exponential drift, resulting in a set of recurrence equations. We explicitly solve the expansion up to second order and comment on higher-order solutions. The first-passage-time density function arises naturally from the survival probability and preserves the proposed expansion. Explicit results, related properties, and limit behaviors are analyzed and extensively compared to numerical simulations.

I. INTRODUCTION

Since the primeval discovery of the Brownian motion and its statistical analysis, the variety of applications in which a relevant stochastic description result is obtained is steadily increasing [1, 2]. The Wiener process and the Ornstein-Uhlenbeck process are idealized statistical descriptions that apply to many systems. One of the most valuable theoretical tools to deal with these and other stochastic processes is the Fokker-Planck (FP) formalism [3, 4]. In this framework different realizations of a system are described by the probability density to find the system in a given state at a certain time, and a diffusion equation describes its temporal evolution. Two related questions of wide interest in several areas are the probability that the system remains in a certain domain at a given time and the instant at which the system leaves it for the first time. Given the stochastic nature of the process, different realizations of the system leave this survival domain at different times and it is natural to consider what the statistical properties of this random variable are. This question constitutes the so-called first-passage-time (FPT) problem [1, 3–6].

The survival probability as well as the FPT problem is easy to formulate but difficult to tackle, except for some simple cases. In particular, for a Wiener process driven by a constant and positive drift toward a fixed positive boundary, these quantities have a simple analytical solution [3–8]. However, the extension to a time-dependent drift is not straightforward, mainly because the system is no longer time-homogeneous. In one-dimensional systems, the main work on this topic possibly is [9]. In that study, the author extended, via the forward FP description, the classical results of Siegert [6] for a particle

being driven by a *small* time-dependent potential, superimposed on a general field. By applying a perturbation scheme, the author derives the recurrence relations between the linear corrections of the moments of the FPT density function. Other series of works have analyzed the behavior of the system in a time-dependent sinusoidal drift, in general, studied in the context of stochastic resonance (see [10–13] for seminal works on this topic for the Wiener process with an absorbing boundary; for other processes, we refer the reader to [14]). However, we are interested in the FPT problem of the Wiener process driven by an exponential time-dependent drift because it naturally arises in neuroscience, when modeling spiking neurons with adaptation currents [15]. This process can also be used to model a neuron with an exponential time-dependent threshold (see [16] for the transformation between an exponential time-dependent drift to an exponential time-dependent threshold). With reference to moving thresholds, the main related work is [17], where the authors analyzed the moments of the FPT density function of a Markov process with a moving barrier, giving some specific examples applied to biological sciences.

In this work, we study the survival probability and the FPT density function of the described system in the framework of the backward FP formalism. We describe the complete statistics, instead of focusing on its moments as in previous studies. We obtain the equation and conditions governing the survival probability and propose a solution in terms of an expansion in the exponential drift intensity. This results in an infinite set of recurrence equations, which we explicitly solve up to second order. Higher-order terms are outlined and discussed. In particular, we show that all order functions exist and depend exclusively on the actual time difference when the initial conditions are imposed for the backward state, as physical considerations require. This constitutes the exact solution of the problem. From the knowledge of the survival probability it is straightforward to derive

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the *complete* FPT density function, which in turn results in an expansion series. Since it is natural to solve the equations via a Laplace transformation, we review some important related properties easy to compute from the Laplace transform of the FPT density function.

In the second part of this work, we focus on the explicit results we have obtained and compare them with numerical simulations. Given that truncation of the expansion series results in an approximate solution, we mostly analyze the system in the linear regime. Related properties and the behavior of the linear solution in different limits are also considered.

II. THEORY

In one dimension, the FPT problem can be basically formulated as follows: a state variable evolves stochastically according to a given law in its phase space, and we are interested in describing when this variable leaves a certain domain for the first time. To deal with this problem a number of different methods or approaches had been described, mostly based on the knowledge of the time-dependent probability density or its temporal evolution [3–7]. Here, we first solve the survival probability in terms of the backward FP equation and then derive the FPT density.

A. Survival probability

The nonautonomous system we address here can be described in terms of the Langevin equation,

$$\frac{dx}{dt} = \mu + \frac{\epsilon}{\tau_d} e^{-(t-t_0)/\tau_d} + \xi(t), \quad (1)$$

where x is the state variable (position, voltage, etc.; hereafter, the position), t is the time, μ is the constant part of the drift, ϵ quantifies the strength of the exponential time-dependent drift with time constant τ_d , and $\xi(t)$ is a Gaussian white noise characterized by $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(t') \rangle = 2D \delta(t-t')$, with D as a constant.

Suppose we have a particle at position x_0 at time t_0 and it evolves to a position x' at a posterior time t' ($t' > t_0$) according to a transition probability density $P(x', t'|x_0, t_0)$. Clearly, in a FPT problem a certain region of the domain is forbidden and actually the transition probability density has implicitly incorporated this fact. In this work, we analyze the region defined by a constant boundary, $x' < x_{\text{thr}}$, which set the survival domain, so the forbidden region is $x \geq x_{\text{thr}}$.

The survival probability $F(t'|x_0, t_0)$ is the probability that the particle remains in the survival domain at time t' given the initial conditions, and it is given simply by integration of the transition probability in the x' domain

$$F(t'|x_0, t_0) = \int_{-\infty}^{x_{\text{thr}}} P(x', t'|x_0, t_0) dx'. \quad (2)$$

To describe the transition probability density we use the backward FP equation. In this case, given that the state variable is at position x' at time t' , the probability density of the particle being at the position x at an earlier time t ($t < t'$) is given by

$$\begin{aligned} \frac{\partial P(x', t'|x, t)}{\partial t} = & - \left[\mu + \frac{\epsilon}{\tau_d} e^{-(t-t_0)/\tau_d} \right] \frac{\partial P(x', t'|x, t)}{\partial x} \\ & - D \frac{\partial^2 P(x', t'|x, t)}{\partial x^2}. \end{aligned} \quad (3)$$

The drift coefficient quantifies the first moment of the differential transition density in the neighborhood of the backward state (x, t) . Necessarily, the local level of the exponential term is relative to the initial time t_0 , breaking up time homogeneity.

The FPT is incorporated with the initial and boundary conditions: $P(x', t'|x, t = t') = 1$ for $x < x_{\text{thr}}$ and 0 for $x \geq x_{\text{thr}}$, and $P(x', t'|x = x_{\text{thr}}, t) = 0$.

Integration of Eq. (3) in x' from $-\infty$ to x_{thr} yields the survival probability from time t to time t' :

$$\begin{aligned} \frac{\partial F(t'|x, t)}{\partial t} = & - \left[\mu + \frac{\epsilon}{\tau_d} e^{-(t-t_0)/\tau_d} \right] \frac{\partial F(t'|x, t)}{\partial x} \\ & - D \frac{\partial^2 F(t'|x, t)}{\partial x^2}. \end{aligned} \quad (4)$$

Since t' is a parameter, we make the substitution $\tau = t' - t$ and rename the probability $F(x, \tau; t')$. The corresponding equation is

$$\begin{aligned} \frac{\partial F(x, \tau; t')}{\partial \tau} = & \left[\mu + \frac{\epsilon}{\tau_d} e^{-(t'-t_0)/\tau_d} e^{\tau/\tau_d} \right] \frac{\partial F(x, \tau; t')}{\partial x} \\ & + D \frac{\partial^2 F(x, \tau; t')}{\partial x^2}, \end{aligned} \quad (5)$$

with $F(x, \tau = 0; t') = 1$ for $x < x_{\text{thr}}$ and 0 for $x \geq x_{\text{thr}}$, and $F(x = x_{\text{thr}}, \tau; t') = 0$.

To solve this equation we propose an expansion in powers of ϵ :

$$\begin{aligned} F(x, \tau; t') &= F_0(x, \tau; t') + \epsilon F_1(x, \tau; t') + \epsilon^2 F_2(x, \tau; t') + \dots \\ &= \sum_{n=0}^{\infty} \epsilon^n F_n(x, \tau; t'). \end{aligned} \quad (6)$$

Replacing Eq. (6) into Eq. (5) and grouping in orders of ϵ , we obtain

$$\begin{aligned}
& \left[\frac{\partial F_0}{\partial \tau} - \mu \frac{\partial F_0}{\partial x} - D \frac{\partial^2 F_0}{\partial x^2} \right] \\
& + \sum_{n=1}^{\infty} \epsilon^n \left[\frac{\partial F_n}{\partial \tau} - \mu \frac{\partial F_n}{\partial x} - \frac{1}{\tau_d} e^{-(t'-t_0)/\tau_d} e^{\tau/\tau_d} \frac{\partial F_{n-1}}{\partial x} \right. \\
& \quad \left. - D \frac{\partial^2 F_n}{\partial x^2} \right] = 0, \quad (7)
\end{aligned}$$

where we have simplified the notation for the sake of clarity.

Since ϵ is a parameter, each term in brackets should be identically 0. Therefore, to find the survival probability we have to solve

$$\frac{\partial F_0}{\partial \tau} - \mu \frac{\partial F_0}{\partial x} - D \frac{\partial^2 F_0}{\partial x^2} = 0 \quad (8)$$

$$\begin{aligned}
& \frac{\partial F_n}{\partial \tau} - \mu \frac{\partial F_n}{\partial x} - D \frac{\partial^2 F_n}{\partial x^2} = \\
& \quad \frac{1}{\tau_d} e^{-(t'-t_0)/\tau_d} e^{\tau/\tau_d} \frac{\partial F_{n-1}}{\partial x} \quad \text{for } n \geq 1. \quad (9)
\end{aligned}$$

This system of Eqs. (8) and (9), can be solved recursively up to the degree of accuracy needed. To complete the solution of the survival probability we have to define the initial and boundary conditions for all the functions $F_n(x, \tau; t')$, for $n \geq 0$. Again, given the arbitrariness of ϵ , the nonhomogeneous conditions should be imposed to the zeroth-order function. Therefore, initial conditions are

$$F_0(x, \tau = 0; t') = \begin{cases} 1 & \text{if } x < x_{\text{thr}}, \\ 0 & \text{if } x \geq x_{\text{thr}}, \end{cases} \quad (10)$$

$$F_n(x, \tau = 0; t') = 0 \quad \text{for } n \geq 1, \quad (11)$$

whereas boundary condition is $F_n(x = x_{\text{thr}}, \tau; t') = 0$ for $n \geq 0$.

Next we solve the expansion up to the second-order term and analyze higher orders.

1. Zeroth order solution

The system described by Eqs. (8) and (10) corresponds to the constant drift case ($\epsilon = 0$). The survival probability of the Wiener process with constant drift and diffusion coefficients is a time-homogeneous process (the system remains unchanged with a shift in t') and easy to solve in Laplace domain. Omitting the dependence in s (to solve the equation, s acts as a parameter), this probability reads

$$\tilde{F}_0^L(x) = \frac{1}{s} - \frac{1}{s} \exp \left\{ \frac{(x_{\text{thr}} - x)}{2D} \left[\mu - \sqrt{\mu^2 + 4Ds} \right] \right\}, \quad (12)$$

where we denote $\tilde{F}_0^L(x)$ the Laplace transform of $F_0(x, \tau)$ to the s domain (due to time homogeneity, t' only appears in τ). In deriving Eq. (12) we have used the fact that $\tilde{F}_0^L(x \rightarrow -\infty)$ is bounded.

By the inverse Laplace transformation of Eq. (12), we obtain the solution in terms of $\tau = t' - t$, $F_0(x, t' - t)$. At this point we state the initial conditions of the problem, $x = x_0$ and $t = t_0$. Therefore, $F(x, t' - t) \rightarrow F(x_0, t' - t_0)$. Again, replacing $\tau = t' - t_0$ (now, τ is the actual time difference) and transforming back to the s domain, we obtain

$$\tilde{F}_0^L(s) = \frac{1}{s} - \frac{1}{s} \exp \left\{ \frac{(x_{\text{thr}} - x_0)}{2D} \left[\mu - \sqrt{\mu^2 + 4Ds} \right] \right\}, \quad (13)$$

where now we have recovered the dependence on s in the notation. By comparing Eqs. (12) and (13) we note just a single change, $x \rightarrow x_0$. However, the procedure described is important in time-inhomogeneous problems and it will be important when solving the following orders.

Even when the inverse Laplace transform of Eq. (13) is available, we disregard this step since, as we will see later when deriving the FPT density function, it is unnecessary (and actually it is related to the FPT cumulative distribution).

2. First order solution

The first order term is given by the solution of Eq. (9) for $n = 1$,

$$\begin{aligned}
& \frac{\partial F_1(x, \tau; t')}{\partial \tau} - \mu \frac{\partial F_1(x, \tau; t')}{\partial x} - D \frac{\partial^2 F_1(x, \tau; t')}{\partial x^2} = \\
& \quad \frac{1}{\tau_d} e^{-(t'-t_0)/\tau_d} \frac{\partial}{\partial x} \left[e^{\tau/\tau_d} F_0(x, \tau; t') \right], \quad (14)
\end{aligned}$$

with the corresponding initial and boundary conditions.

This equation can be solved via the Laplace transform. In the s domain, Eq. (14) reads

$$\begin{aligned}
& s \tilde{F}_1^L(x; t') - \mu \frac{d\tilde{F}_1^L(x; t')}{dx} - D \frac{d^2 \tilde{F}_1^L(x; t')}{dx^2} = \\
& \quad \frac{1}{\tau_d} e^{-(t'-t_0)/\tau_d} \frac{d}{dx} \left\{ \mathcal{L} [F_0(x, \tau; t')]_{(s)} \right\}_{|s=1/\tau_d}, \quad (15)
\end{aligned}$$

where $\mathcal{L}[\cdot]_{(s)}$ represents the Laplace transform operator and $\tilde{F}_1^L(x; t')$ is the Laplace transform of $F_1(x, \tau; t')$. Substituting the result we obtained before, Eq. (12) (note that the initial state of the problem is *not* already evaluated), into Eq. (15) we have

$$s \tilde{F}_1^L(x; t') - \mu \frac{d\tilde{F}_1^L(x; t')}{dx} - D \frac{d^2\tilde{F}_1^L(x; t')}{dx^2} = \frac{1}{\tau_d} e^{-(t'-t_0)/\tau_d} \frac{\left[\mu - \sqrt{\mu^2 + 4D(s - 1/\tau_d)} \right]}{2D(s - 1/\tau_d)} \cdot \exp \left\{ \frac{(x_{\text{thr}} - x)}{2D} \left[\mu - \sqrt{\mu^2 + 4D(s - 1/\tau_d)} \right] \right\}. \quad (16)$$

The general solution to this equation is given by

$$\begin{aligned} \tilde{F}_1^L(x; t') = & C_1 \exp \left(-\frac{\mu + \sqrt{\mu^2 + 4Ds}}{2D} x \right) \\ & + C_2 \exp \left(-\frac{\mu - \sqrt{\mu^2 + 4Ds}}{2D} x \right) \\ & + \frac{1}{2D} e^{-(t'-t_0)/\tau_d} \frac{\left[\mu - \sqrt{\mu^2 + 4D(s - 1/\tau_d)} \right]}{(s - 1/\tau_d)} \\ & \cdot \exp \left\{ \frac{(x_{\text{thr}} - x)}{2D} \left[\mu - \sqrt{\mu^2 + 4D(s - 1/\tau_d)} \right] \right\}, \quad (17) \end{aligned}$$

valid for $\text{Re}(s) \geq 1/\tau_d$.

Taking into account that $\tilde{F}_1^L(x \rightarrow -\infty; t')$ is bounded and the boundary condition is $\tilde{F}_1^L(x = x_{\text{thr}}; t') = 0$, we obtain

$$\begin{aligned} \tilde{F}_1^L(x; t') = & \frac{1}{2D} e^{-(t'-t_0)/\tau_d} \frac{\left[\mu - \sqrt{\mu^2 + 4D(s - 1/\tau_d)} \right]}{(s - 1/\tau_d)} \\ & \cdot \left\{ \exp \left\{ \frac{(x_{\text{thr}} - x)}{2D} \left[\mu - \sqrt{\mu^2 + 4D(s - 1/\tau_d)} \right] \right\} \right. \\ & \left. - \exp \left\{ \frac{(x_{\text{thr}} - x)}{2D} \left[\mu - \sqrt{\mu^2 + 4Ds} \right] \right\} \right\}. \quad (18) \end{aligned}$$

We further operate with the inverse Laplace transform of Eq. (18), which is

$$\begin{aligned} F_1(x, \tau; t') = & \frac{1}{2D} e^{-(t'-t_0)/\tau_d} \\ & \cdot \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} e^{s\tau} \frac{\left[\mu - \sqrt{\mu^2 + 4D(s - 1/\tau_d)} \right]}{(s - 1/\tau_d)} \\ & \cdot \left\{ \exp \left\{ \frac{(x_{\text{thr}} - x)}{2D} \left[\mu - \sqrt{\mu^2 + 4D(s - 1/\tau_d)} \right] \right\} \right. \\ & \left. - \exp \left\{ \frac{(x_{\text{thr}} - x)}{2D} \left[\mu - \sqrt{\mu^2 + 4Ds} \right] \right\} \right\} ds, \quad (19) \end{aligned}$$

where j represents the imaginary unit and $\sigma \geq 1/\tau_d$. Taking the substitution $z = s - 1/\tau_d$, we obtain

$$\begin{aligned} F_1(x, \tau; t') = & \frac{1}{2D} e^{-(t'-t_0)/\tau_d} e^{\tau/\tau_d} \\ & \cdot \frac{1}{2\pi j} \int_{\sigma_z-j\infty}^{\sigma_z+j\infty} e^{z\tau} \frac{\left[\sqrt{\mu^2 + 4Dz} - \mu \right]}{z} \\ & \cdot \left\{ \exp \left\{ \frac{(x_{\text{thr}} - x)}{2D} \left[\mu - \sqrt{\mu^2 + 4D(z + 1/\tau_d)} \right] \right\} \right. \\ & \left. - \exp \left\{ \frac{(x_{\text{thr}} - x)}{2D} \left[\mu - \sqrt{\mu^2 + 4Dz} \right] \right\} \right\} dz, \quad (20) \end{aligned}$$

where now, it is easy to check that the region of convergence of the integrand is $\text{Re}(z) = \sigma_z \geq 0$. However, there still is an exponential factor that makes the expression diverge.

At this point we are able to evaluate the real conditions of the problem: $x = x_0$ and $t = t_0$. Remembering that $\tau = t' - t$, the latter condition imposes that the two exponential factors (before the integral) in Eq. (20) cancel each other. Hereafter, we use τ to represent the actual time referred to the initial time, $\tau = t' - t_0$. Taking the Laplace transform on this variable, from Eq. (20) the function $F_1(\tau)$ (note that x was evaluated and the dependence on t' is exclusively given by the combination in τ) transforms to

$$\begin{aligned} \tilde{F}_1^L(s) = & \frac{1}{2D} \frac{\left[\sqrt{\mu^2 + 4Ds} - \mu \right]}{s} \\ & \cdot \left\{ \exp \left\{ \frac{(x_{\text{thr}} - x_0)}{2D} \left[\mu - \sqrt{\mu^2 + 4D(s + 1/\tau_d)} \right] \right\} \right. \\ & \left. - \exp \left\{ \frac{(x_{\text{thr}} - x_0)}{2D} \left[\mu - \sqrt{\mu^2 + 4Ds} \right] \right\} \right\}, \quad (21) \end{aligned}$$

valid for $\text{Re}(s) \geq 0$.

3. A note on the higher-order solutions

In this subsection we remark on some aspects of the existence and the convergence of higher-order terms expressed in the Laplace domain. The higher-order terms in the expansion, Eq. (6), correspond to the solution of Eq. (9) with the appropriate initial and boundary conditions ($n \geq 2$). In particular, we obtain an equation analogous to Eq. (14):

$$\begin{aligned} \frac{\partial F_n(x, \tau; t')}{\partial \tau} - \mu \frac{\partial F_n(x, \tau; t')}{\partial x} - D \frac{\partial^2 F_n(x, \tau; t')}{\partial x^2} = \\ \frac{1}{\tau_d} e^{-(t'-t_0)/\tau_d} \frac{\partial}{\partial x} \left[e^{\tau/\tau_d} F_{n-1}(x, \tau; t') \right]. \quad (22) \end{aligned}$$

The term on the right-hand side of the equation corresponds to a source because $F_{n-1}(x, \tau; t')$ was already

solved. As in the first-order case, the knowledge of the source term in the s domain enables us to readily Laplace transform the equation, obtaining an ordinary differential equation with a forcing term. The homogeneous part of the solution is exactly the same as that in Eq. (17) (terms with unknown constants C_i) and the particular solution is different for different orders. Moreover, C_1 has to be 0 for bounded solutions and the existence of the particular solution is given as a sum of the exponential factors present in $\tilde{F}_{n-1}^L(x; t')|_{s=1/\tau_d}$. After evaluation of the boundary condition, the solution is given as the sum of $n + 1$ exponential terms.

Here we note the structure that this forcing term imposes on the solution. Since Eq. (22) operates in the backward state (x, t) , the previous-order solution $F_{n-1}(x, \tau; t')$ must not be evaluated in the initial state (x_0, t_0) , or correspondingly, its Laplace transform should be done for the variable $\tau = t' - t$ and not for $\tau = t' - t_0$.

To simplify, we exemplify the concepts of convergence with the second-order solution and then extend the conclusion to all orders. In the Laplace domain, the equation governing the second-order solution is

$$s \tilde{F}_2^L(x; t') - \mu \frac{d\tilde{F}_2^L(x; t')}{dx} - D \frac{d^2\tilde{F}_2^L(x; t')}{dx^2} = \frac{1}{\tau_d} e^{-(t'-t_0)/\tau_d} \frac{d}{dx} \left\{ \mathcal{L}[F_1(x, \tau; t')]_{(s)} \right\} |_{s=1/\tau_d}, \quad (23)$$

where the Laplace transform of the previous-order solution is given by Eq. (18). Prior to the evaluation of the initial state, this solution has a region of convergence $\text{Re}(s) \geq 1/\tau_d$. It is easy to check that, due to the delay introduced in the Laplace domain, the forcing term in Eq. (23) will impose that the region of convergence of the Laplace transform of the second-order solution is $\text{Re}(s) \geq 2/\tau_d$, but now a factor $\exp[-2(t' - t_0)/\tau_d]$ appears. Explicitly, the equation governing the second-order is given by

$$s \tilde{F}_2^L(x; t') - \mu \frac{d\tilde{F}_2^L(x; t')}{dx} - D \frac{d^2\tilde{F}_2^L(x; t')}{dx^2} = \frac{1}{2D\tau_d} e^{-2(t'-t_0)/\tau_d} \frac{\left[\mu - \sqrt{\mu^2 + 4D(s - 2/\tau_d)} \right]}{2D(s - 2/\tau_d)} \cdot \left\{ \left[\mu - \sqrt{\mu^2 + 4D(s - 1/\tau_d)} \right] \cdot \exp \left\{ \frac{(x_{\text{thr}} - x)}{2D} \left[\mu - \sqrt{\mu^2 + 4D(s - 1/\tau_d)} \right] \right\} - \left[\mu - \sqrt{\mu^2 + 4D(s - 2/\tau_d)} \right] \cdot \exp \left\{ \frac{(x_{\text{thr}} - x)}{2D} \left[\mu - \sqrt{\mu^2 + 4D(s - 2/\tau_d)} \right] \right\} \right\}, \quad (24)$$

and its solution is

$$\tilde{F}_2^L(x; t') = \frac{1}{2D} e^{-2(t'-t_0)/\tau_d} \frac{\left[\mu - \sqrt{\mu^2 + 4D(s - 2/\tau_d)} \right]}{2D(s - 2/\tau_d)} \cdot \sum_{i=0}^2 a_i \exp \left\{ \frac{(x_{\text{thr}} - x)}{2D} \left[\mu - \sqrt{\mu^2 + 4D(s - i/\tau_d)} \right] \right\}, \quad (25)$$

where

$$\begin{aligned} a_0 &= \sqrt{\mu^2 + 4D(s - 1/\tau_d)} - \frac{1}{2} \sqrt{\mu^2 + 4D(s - 2/\tau_d)} \\ &\quad - \frac{1}{2} \mu, \\ a_1 &= \mu - \sqrt{\mu^2 + 4D(s - 1/\tau_d)}, \\ a_2 &= -\frac{1}{2} \left[\mu - \sqrt{\mu^2 + 4D(s - 2/\tau_d)} \right]. \end{aligned} \quad (26)$$

Proceeding as in Eqs. (19) and (20), we obtain two exponential factors, $\exp[-2(t' - t_0)/\tau_d]$ and $\exp(2\tau/\tau_d)$, that cancel each other when the initial state (x_0, t_0) is imposed. This cancellation actually means that the second-order term of the survival probability, with the initial state imposed, depends on time exclusively through the combination $\tau = t' - t_0$ (actual time difference). Therefore, its Laplace transform (on the variable $\tau = t' - t_0$) with the initial state evaluated is

$$\tilde{F}_2^L(s) = \frac{1}{2D} \frac{\left[\mu - \sqrt{\mu^2 + 4Ds} \right]}{2Ds} \sum_{i=0}^2 b_i(s) \cdot \exp \left\{ \frac{(x_{\text{thr}} - x_0)}{2D} \left[\mu - \sqrt{\mu^2 + 4D(s + i/\tau_d)} \right] \right\}, \quad (27)$$

where

$$\begin{aligned} b_0(s) &= -\frac{1}{2} \left[\mu - \sqrt{\mu^2 + 4Ds} \right], \\ b_1(s) &= \mu - \sqrt{\mu^2 + 4D(s + 1/\tau_d)}, \\ b_2(s) &= \sqrt{\mu^2 + 4D(s + 1/\tau_d)} - \frac{1}{2} \sqrt{\mu^2 + 4Ds} \\ &\quad - \frac{1}{2} \mu. \end{aligned} \quad (28)$$

Recursively, the n th-order solution in the backward state (x, t) has a Laplace transform valid for $\text{Re}(s) \geq n/\tau_d$ with a factor $\exp[-n(t' - t_0)/\tau_d]$. Therefore, the preceding conclusion extends to all orders.

Given the existence of the solution of all terms, the expansion proposed in Eq. (6) constitutes the exact solution of the system.

B. First-passage time density

In the previous subsection, we demonstrate that the expansion given by Eq. (6), with the initial state evaluated, (x_0, t_0) , actually reads

$$\begin{aligned}
F(x_0, \tau) &= F_0(x_0, \tau) + \epsilon F_1(x_0, \tau) + \epsilon^2 F_2(x_0, \tau) + \dots \\
&= \sum_{n=0}^{\infty} \epsilon^n F_n(x_0, \tau),
\end{aligned} \tag{29}$$

where the dependence on time appears exclusively through the combination $\tau = t' - t_0$.

Once the initial conditions are stated, by definition, $F(x_0, \tau)$ is the probability that the particle remains at time $\tau = t' - t_0$ in the survival domain and, hence, equals the probability that the FPT is posterior to τ : $F(x_0, \tau) = \text{Prob}(T > \tau)$, where T represents the FPT. In terms of the cumulative distribution function of the FPT random variable, $\Phi(\tau)$, this means that $F(x_0, \tau) = 1 - \Phi(\tau)$ (hereafter, x_0 is a parameter and can be disregarded from notation). The density function, $\phi(\tau)$, is given by

$$\phi(\tau) = \frac{d\Phi(\tau)}{d\tau} = -\frac{\partial F(x_0, \tau)}{\partial \tau}, \tag{30}$$

which means that the FPT density function has an expansion given by

$$\phi(\tau) = -\sum_{n=0}^{\infty} \epsilon^n \frac{\partial F_n(x_0, \tau)}{\partial \tau}. \tag{31}$$

Remembering that the initial condition in the diffusion problem reads $F(x, \tau = 0; t') = 1$ for $x < x_{\text{thr}}$, it results that $F(x_0, \tau = 0) = 1$ in the solution already evaluated with the conditions of the problem (and obviously $x_0 < x_{\text{thr}}$ for a nontrivial problem). Therefore, the Laplace transform of Eq. (30) reads

$$\tilde{\phi}^L(s) = 1 - s \tilde{F}^L(x_0, s), \tag{32}$$

where $\tilde{\phi}^L(s)$ [$\tilde{F}^L(x_0, s)$] is the Laplace transform of $\phi(\tau)$ [$F(x_0, \tau)$].

Equivalently, in terms of the expansion for $\tilde{F}^L(x_0, s)$ [see Eq. (29)], the Laplace transform of the density is

$$\tilde{\phi}^L(s) = 1 - s \sum_{n=0}^{\infty} \epsilon^n \tilde{F}_n^L(x_0, s), \tag{33}$$

where $\tilde{F}_n^L(x_0, s)$ is the Laplace transform of the n th term in the expansion of $F(x_0, \tau)$, $F_n(x_0, \tau)$.

Since $\phi(\tau)$ has an expansion given by Eq. (31), it is natural to write

$$\phi(\tau) = \sum_{n=0}^{\infty} \epsilon^n \phi_n(\tau), \tag{34}$$

where

$$\phi_n(\tau) = -\frac{\partial F_n(x_0, \tau)}{\partial \tau}. \tag{35}$$

In the s domain, Eq. (34) reads

$$\tilde{\phi}^L(s) = \sum_{n=0}^{\infty} \epsilon^n \tilde{\phi}_n^L(s), \tag{36}$$

where $\tilde{\phi}_n^L(s)$ is the Laplace transform of the n th term in the expansion of $\phi(\tau)$, $\phi_n(\tau)$, and it is given by

$$\tilde{\phi}_0^L(s) = 1 - s \tilde{F}_0^L(x_0, s), \tag{37}$$

$$\tilde{\phi}_n^L(s) = -s \tilde{F}_n^L(x_0, s), \text{ for } n \geq 1. \tag{38}$$

For example, from the findings in the previous subsection, the terms in the expansion up to the first order of the FPT density function are

$$\tilde{\phi}_0^L(s) = \exp \left\{ \frac{(x_{\text{thr}} - x_0)}{2D} \left[\mu - \sqrt{\mu^2 + 4Ds} \right] \right\}, \tag{39}$$

$$\begin{aligned}
\tilde{\phi}_1^L(s) &= \frac{\left[\mu - \sqrt{\mu^2 + 4Ds} \right]}{2D} \\
&\cdot \left\{ \exp \left\{ \frac{(x_{\text{thr}} - x_0)}{2D} \left[\mu - \sqrt{\mu^2 + 4D(s + 1/\tau_d)} \right] \right\} \right. \\
&\quad \left. - \exp \left\{ \frac{(x_{\text{thr}} - x_0)}{2D} \left[\mu - \sqrt{\mu^2 + 4Ds} \right] \right\} \right\}.
\end{aligned} \tag{40}$$

Eq. (39) is the classical result for the FPT problem with constant drift μ and diffusion D coefficients [7], consistent with our approach.

C. Related properties of the first-passage time density

Since the solution of the proposed expansion is naturally obtained in the Laplace domain, here we review some properties easy to calculate from this knowledge. It is easy to check that the moments of the density function satisfy

$$\langle \tau^k \rangle = \int_0^{\infty} \phi(\tau) \tau^k d\tau = (-1)^k \frac{d^k \tilde{\phi}^L(s)}{ds^k} \Big|_{s=0}, \tag{41}$$

which means that all the moments preserve the expansion in ϵ

$$\langle \tau^k \rangle = \sum_{n=0}^{\infty} \epsilon^n \langle \tau^k \rangle_{\phi_n}, \tag{42}$$

where

$$\langle \tau^k \rangle_{\phi_n} = (-1)^k \frac{d^k \tilde{\phi}_n^L(s)}{ds^k} \Big|_{s=0}. \quad (43)$$

For example, the first two moments for the unperturbed case ($n = 0$) are

$$\begin{aligned} \langle \tau \rangle_{\phi_0} &= \frac{x_{\text{thr}} - x_0}{\mu}, \\ \langle \tau^2 \rangle_{\phi_0} &= \frac{2D(x_{\text{thr}} - x_0)}{\mu^3} + \frac{(x_{\text{thr}} - x_0)^2}{\mu^2}, \end{aligned} \quad (44)$$

which correspond to the constant drift case [7].

The linear changes in these properties, Eq. (42), for $n = 1$, are

$$\begin{aligned} \langle \tau \rangle_{\phi_1} &= \frac{1}{\mu} \\ &\cdot \left\{ \exp \left[\frac{(x_{\text{thr}} - x_0)}{2D} \left(\mu - \sqrt{\mu^2 + 4D/\tau_d} \right) \right] - 1 \right\}, \\ \langle \tau^2 \rangle_{\phi_1} &= \frac{2}{\mu^2} \\ &\cdot \left\{ \left[\frac{\mu(x_{\text{thr}} - x_0)}{\sqrt{\mu^2 + 4D/\tau_d}} + \frac{D}{\mu} \right] \right. \\ &\cdot \exp \left[\frac{(x_{\text{thr}} - x_0)}{2D} \left(\mu - \sqrt{\mu^2 + 4D/\tau_d} \right) \right] \\ &\left. - (x_{\text{thr}} - x_0) - \frac{D}{\mu} \right\}. \end{aligned} \quad (45)$$

These results, Eq. (45), coincide with those of the corresponding case in [9], obtained from a different approach.

The assessment of the complete density function in the Laplace domain enables us to obtain another important property. The successive ordering of FPTs of Wiener processes, each of them independent of the history (in our system, this means the fixed escape domain and initial state), constitutes a renewal process. Given the times $\{t_k\}$ when the system reaches the threshold x_{thr} starting from x_0 and setting it again to x_0 , we can construct a “spike train”, $X(t)$, defined by

$$X(t) = \sum_{\{t_k\}} \delta(t - t_k), \quad (46)$$

representing a renewal point process.

The Fourier transform of $X(t)$ is the spike train spectral density $S(\omega)$, which represents an important property in some fields, such as neuroscience [22]. It is related to the density function of a *single* escape process, expressed in the Laplace domain, through [21, 22]

$$S(\omega) = \frac{1}{2\pi \langle \tau \rangle} \left[1 + \frac{\tilde{\phi}^L(j\omega)}{1 - \tilde{\phi}^L(j\omega)} + \frac{\tilde{\phi}^L(-j\omega)}{1 - \tilde{\phi}^L(-j\omega)} \right], \quad (47)$$

where ω is the angular frequency. The sequence of the renewal times is mean subtracted; otherwise, a δ peak appears at frequency 0.

In particular, the constant driving case ($\epsilon = 0$) is analytically tractable and a relatively simple expression is found in [22]. The exponential driving case ($\epsilon \neq 0$) corresponds to the spike train produced by a perfect integrate-and-fire neuron with an exponential time-dependent threshold [16].

III. COMPARISON TO NUMERICAL RESULTS

In this section we test different theoretical results and compare them with numerical simulations. As shown, the expansion given by Eq. (6) is the exact solution of the system. However, the explicit computation of successive terms in the expansion is performed up to certain order. Since truncation introduces an error for any finite order, we mainly focus on the first-order expansion with small values of ϵ . In this case, the time-dependent exponential drift can be thought of as a perturbation to the unperturbed system defined by $\epsilon = 0$ (constant drift case). Without mathematical loss, we set all quantities of the system to nondimensional units.

A. Linear order expansion

In Figs. 1(a) and 1(b) we show the FPT density obtained from simulations for different intensities of the exponential drift [(a) $\epsilon = -0.5$ and (b) $\epsilon = -2.0$]. The histogram obtained from simulations (stair-like solid line) is compared with different predictions. The zeroth-order prediction, $\phi_0(\tau)$, is given by the inverse Laplace transform of Eq. (39)

$$\phi_0(\tau) = \frac{(x_{\text{thr}} - x_0)}{\sqrt{4\pi D\tau^3}} \exp \left\{ -\frac{[(x_{\text{thr}} - x_0) - \mu\tau]^2}{4D\tau} \right\}, \quad (48)$$

and corresponds to the constant drift case ($\epsilon = 0$). The linear-order solution is composed of $\phi_0(\tau) + \epsilon\phi_1(\tau)$. The function $\phi_1(\tau)$ is obtained from numerical Laplace inversion of Eq. (40).

As shown in Fig. 1(a), for low intensities of the exponential drift, the FPT statistics is well characterized by the linear order. As expected, when the intensity is increased, higher-order effects become significant and the linear expansion is not enough [Fig. 1(b)]. In this case, the second-order solution, given by Eqs. (27) and (38) and numerical inverse Laplace transformation, successfully accounts for the numerical data.

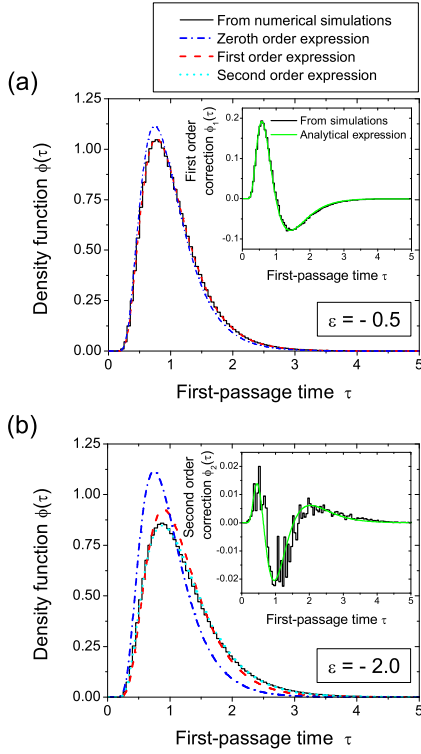


FIG. 1: (Color online) First-passage time density functions $\phi(\tau)$ for different intensities of the exponential time-dependent drift. Histograms obtained from simulations (stair-like solid lines) are compared with analytical results: zeroth-order [dotted-dashed (blue) lines], $\phi_0(\tau)$; first-order [dashed (red) lines], $\phi_0(\tau) + \epsilon\phi_1(\tau)$; and second-order [dotted (cyan) lines], $\phi_0(\tau) + \epsilon\phi_1(\tau) + \epsilon^2\phi_2(\tau)$, expressions. $\phi_0(\tau)$ is given by Eq. (48), whereas $\phi_1(\tau)$ and $\phi_2(\tau)$ are obtained from numerical Laplace inversion of the corresponding expressions [Eq. (40) for $\phi_1(\tau)$ and Eqs. (27) and (38) for $\phi_2(\tau)$]. (a) Low intensity, $\epsilon = -0.5$; (b) high intensity, $\epsilon = -2.0$. Insets: The analytical expression [solid (green) line] for the first-(second-) order function $\phi_1(\tau)$ [$\phi_2(\tau)$] is compared with the empirical linear (second-order) function (stair-like solid lines), see text for definition. Remaining parameters are $N = 10^7$ simulations for each case, $\mu = 1.0$, $D = 0.1$, $x_{\text{thr}} - x_0 = 1.0$, and $\tau_d = 10.0$.

To directly compare the linear correction $\phi_1(\tau)$ with its numerical equivalent, we construct an empirical linear function as follows. From N FPT processes for the constant ($\epsilon = 0$) and time-dependent ($\epsilon \neq 0$) cases, we obtained their histograms, $\phi_{\text{const}}(\tau)$ and $\phi_{\text{timedep.}}(\tau)$, and define $\phi_1^{\text{empir}}(\tau) = [\phi_{\text{timedep.}}(\tau) - \phi_{\text{const}}(\tau)]/\epsilon$. Obviously, $\phi_{\text{const}}(\tau)$ coincides with $\phi_0(\tau)$ (not shown). If the FPT process is dominated by the linear regime, the empirical function so obtained should agree with the analytical result, $\phi_1(\tau)$. As shown in the inset in Fig. 1(a), both functions coincide for a small perturbation $\epsilon = -0.5$ [stair-like line represents the empirical function, whereas the solid (green) line is the analytical expression] and are a relative mismatch for $\epsilon = -2.0$ (not shown). In this

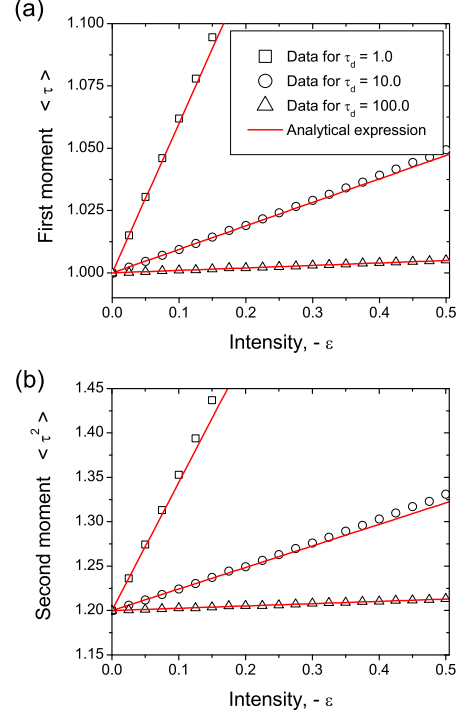


FIG. 2: (Color online) The first two moments as a function of the (negative) intensity of the exponential perturbation, $-\epsilon$, for different τ_d . Analytical expressions [solid (red) line] are given by Eqs. (44) and (45), whereas data obtained from simulations are represented by different symbols. Remaining parameters are as in Fig. 1.

case, given the empirical linear correction constructed with a tiny perturbation $\epsilon = -0.1$, we can construct a similar empirical second-order correction function as $\phi_2^{\text{empir}}(\tau) = [\phi_{\text{timedep.}}(\tau) - \phi_{\text{const}}(\tau) - \epsilon\phi_1^{\text{empir}}(\tau)]/\epsilon^2$. As shown in the inset in Fig. 1(b) this empirical function coincides with its analytic counterpart $\phi_2(\tau)$ (fluctuations due to a finite number of simulations become higher than in the linear construction).

Given the parameters $\mu = 1$ and $(x_{\text{thr}} - x_0) = 1.0$, the mean FPT for the unperturbed case is 1. As shown in the inset in Fig. 1(a), a small additive exponential drift generates a biphasic correction to the constant case density, with a positive weight for times shorter than the unperturbed mean and a negative weight for longer times. The overall effect is to distort the density, increasing the probability of times shorter than the typical time in the unperturbed system leaving the survival domain, for positive perturbations. For negative perturbations, the contrary is true.

B. Moments of the first-passage time density

As expressed by Eq. (42), a linear correction to the FPT density of the unperturbed system is reproduced in

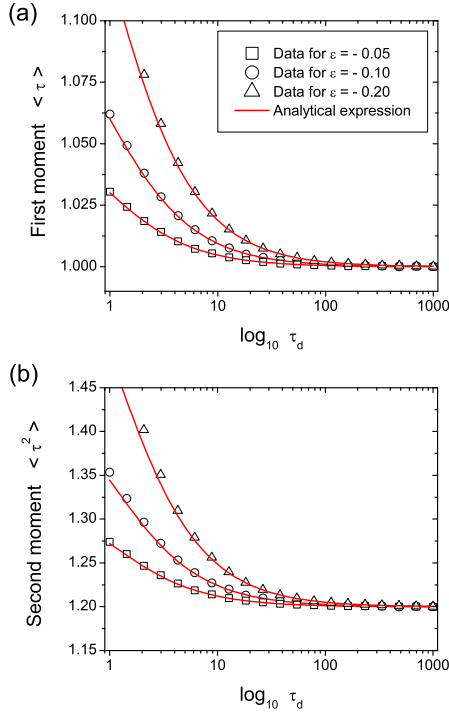


FIG. 3: (Color online) The first two moments as a function of the exponential time scale τ_d (semi-logarithmic plot), for different strengths of the perturbation. Remaining data as in Fig. 2.

all its moments. In Fig. 2 we show the first two moments for the unperturbed case ($\epsilon = 0$) and different strengths of the perturbation, $\epsilon \neq 0$. The analytical expressions for these moments are given by $\langle \tau^k \rangle = \langle \tau^k \rangle_{\phi_0} + \epsilon \langle \tau^k \rangle_{\phi_1}$ ($k = 1, 2$), where $\langle \tau^k \rangle_{\phi_i}$ are given by Eqs. (44) and (45). Arbitrarily, we use negative values for ϵ . In this case, moments shift toward larger values in comparison to the unperturbed case. As shown, both moments [Figs. 2(a) and 2(b)] coincide with the *linear* analytical results for low intensities and mismatch for larger values. To describe these properties properly for large values of ϵ , we should include higher-order terms in the expansion given, Eq. (42). The range of validity of the linear regime is given by the time scale of the exponential perturbation τ_d , which set the effective intensity in the Langevin equation [see Eq. (1)]. As the time scale of the exponential drift increases, the linear regime remains valid over a wider range in the ϵ coordinate.

To stress the preceding paragraph, in Fig. 3 we show the first two moments as a function of the time scale (logarithmic scale) for different perturbation intensities. In this case, a given perturbation ϵ produces a linear distortion for a large time scale τ_d , but higher-order effects become important for smaller time constants. However, as we see in the last subsection, the results we obtained for the linear expansion still hold in the limit of vanishing time scales $\tau_d \rightarrow 0$.

C. Spectral density

To simplify the following discussion, here we set time units in milliseconds, while maintaining x as a nondimensional magnitude. In this case, μ , D , and τ_d are measured as 1/milliseconds, 1/milliseconds, and milliseconds, respectively. As indicated in Sec. II C, the spectral density of a renewal process composed of consecutive first passages (hereafter called the spike train) is easily computed with the Laplace transform of the density function, Eq. (47). For example, the (one-sided) spectral density (per unitary time) [23] of the unperturbed system has a relatively simple expression (see Eq. (3.17) in [22]), which is shown in Fig. 4(a).

As indicated at the beginning of this section, we restrict ourselves to consideration of the effect of a *small* additive exponential time-dependent perturbation on the spectral properties of the spike train evoked by a system driven by a leading constant drift μ . For such a situation, the change in the spectral density of the unperturbed system is hardly noticeable, and therefore, to analyze the frequency-dependent changes introduced by the perturbation, we focus on the ratio between spectral densities. If the unperturbed system is characterized by μ , D and $x_{\text{thr}} - x_0$ [Fig. 4(a)], and an exponential drift is added to the system, defined by ϵ and τ_d , the effect of this perturbation on the spectral density is shown in Fig. 4(b). In this case, we set a negative perturbation $\epsilon < 0$, which implies that the mean FPT increases [see Eq. (45)]. Consequently, the rate of the consecutive first-passage processes decreases to lower frequencies, in comparison with the unperturbed case. This rate is strictly given by the value of the spectral density at infinite (two distant events are uncorrelated, which means a “white” spectrum at frequencies tending to infinity, for stationary processes), $S(\omega \rightarrow \infty)$, and it roughly determines the position of the observed peak [see Fig. 4(a)]. For example, in Fig. 4(a) the rate is defined by $\langle \tau \rangle^{-1} = \mu / (x_{\text{thr}} - x_0) = 1 \text{ ms}^{-1}$. This results in an asymptotic value of 10^3 s^{-1} [a factor equal to 2 appears when considering the *one-sided* spectral density [23], such as that shown in Fig. 4(a)] and a peak located near 10^3 Hz . The lower rate obtained by the addition of a negative exponential perturbation decreases the asymptotic spectral value and shifts the peak to a lower frequency. In the spectral ratio we consider in Fig. 4(b), $S_{\text{pert}}(\omega) / S_{\text{unpert}}(\omega)$, these effects are reflected by an asymptotic value less than unity and a biphasic shape, with a positive (negative) peak located at a lower (higher) rate than the unperturbed rate.

The effects already mentioned (representative of the linear regime) are mainly related to the mean $\langle \tau \rangle$, so we consider an alternative situation where this property does not change. In this case, if the unperturbed system has a given mean [in Fig. 4(a), $\langle \tau \rangle = 1 \text{ ms}$], the perturbed system will be driven by the low exponential time-dependent drift, freely defined by ϵ and τ_d , and the constant component will be modified (in comparison to the unperturbed

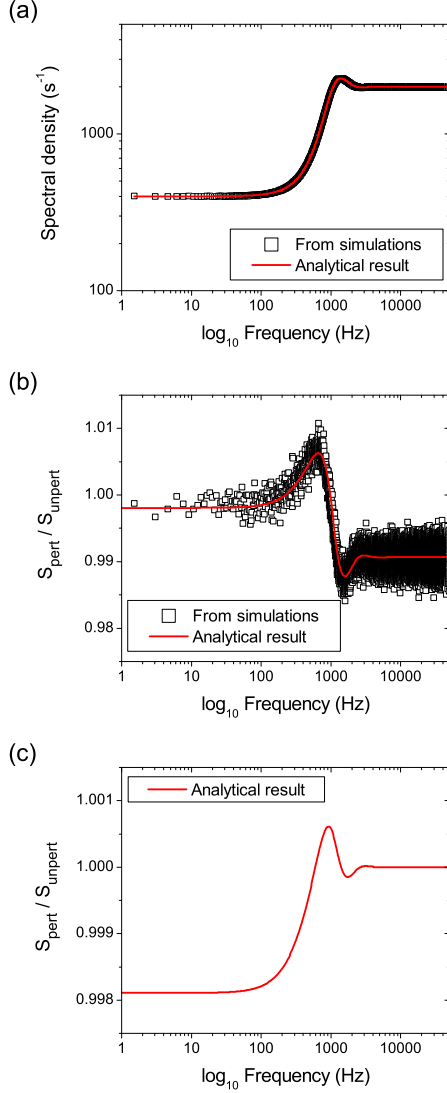


FIG. 4: (Color online) (One-sided) Spectral density (per unitary time) of spike trains for different cases. In each case, the spectral density was obtained as an average of 10^6 independent simulations. (a) Spectral density of the unperturbed case ($\epsilon = 0$). Parameters: $\mu = 1.0 \text{ ms}^{-1}$, $D = 0.1 \text{ ms}^{-1}$, $x_{\text{thr}} - x_0 = 1.0$. (b) Ratio between spectral densities, $S_{\text{pert}}(\omega)/S_{\text{unpert}}(\omega)$, for the exponential time-dependent drift as a perturbation. Same parameters as in (a) for each case, and the perturbed system is additionally defined by $\epsilon = -0.1$ and $\tau_d = 10.0 \text{ ms}$. (c) Ratio between spectral densities, $S_{\text{pert,mc}}(\omega)/S_{\text{unpert}}(\omega)$, for the mean corrected exponential drift (see text). Same parameters as in (a) for the unperturbed case. For the exponential drift case, D and $x_{\text{thr}} - x_0$ are the same as in the unperturbed case, and the exponential drift is characterized by $\epsilon = -0.1$ and $\tau_d = 10.0 \text{ ms}$ [same as in (b)], but the constant component of the drift, $\mu_{\text{pert,mc}}$ is changed in order to obtain the same mean $\langle \tau \rangle$ as in the unperturbed case ($\langle \tau \rangle = 1.0 \text{ ms}$).

system) in order to keep the mean unchanged. In consequence, the asymptotic value for the ratio between the

spectra should be equal to unity, as shown in Fig. 4(c). In this figure we note that other effects are present since the ratio is not flat, but they are an order of magnitude less than the case where the exponential term is a direct additive effect to the unperturbed system [Fig. 4(b)]. Moreover, these effects are hardly noticeable by simulations (even for the large set we used). Therefore, the change in the mean is the main effect introduced by this kind of time-dependent perturbation.

D. Limit behaviors

Finally, here we analyze the behavior of the first-order analytical solution we explicitly obtained, for large and small time scales of the exponential drift.

1. Limit $\tau_d \rightarrow \infty$

In section II we obtained the FPT density function as an expansion in ϵ and explicitly derived the first-order expression [for the FTP density, it is easy to obtain the second-order expression from Eq. (27) via Eq. (38)]. In particular, we have this solution fully characterized in the Laplace domain. To analyze the limit of large time scales, $\tau_d \rightarrow \infty$, we expand Eq. (40) in terms of $(1/\tau_d)$ and keep the lowest terms. Up to order 1, this expansion reads

$$\tilde{\phi}_1^L(s) \approx -\frac{(x_{\text{thr}} - x_0)}{2D\tau_d} \frac{\mu - \sqrt{\mu^2 + 4Ds}}{\sqrt{\mu^2 + 4Ds}} \cdot \exp\left\{\frac{(x_{\text{thr}} - x_0)}{2D} \left[\mu - \sqrt{\mu^2 + 4Ds}\right]\right\}. \quad (49)$$

From this expression we can obtain, by differentiation in s and evaluation at $s = 0$, the linear term in the expansion of the moments of the density function, valid for this limit. The resulting expressions coincide with those in [9].

Eq. (49) shows the Laplace transform of the linear term appearing in the ϵ expansion, for the limit $\tau_d \rightarrow \infty$. In particular, it is possible to find its inverse Laplace transform [20], which reads

$$\begin{aligned} \phi_1^{(\tau_d \rightarrow \infty)}(\tau) &= \frac{(x_{\text{thr}} - x_0)}{\sqrt{4\pi D\tau^3}} \frac{(x_{\text{thr}} - x_0) - \mu\tau}{2D\tau_d} \\ &\cdot \exp\left\{-\frac{[(x_{\text{thr}} - x_0) - \mu\tau]^2}{4D\tau}\right\} \\ &= \frac{(x_{\text{thr}} - x_0) - \mu\tau}{2D\tau_d} \cdot \phi_0(\tau). \end{aligned} \quad (50)$$

In Fig. 5(a) we show the product of the asymptotic linear correction function, Eq. (50), multiplied by τ_d as a function of τ . This product does not depend on τ_d and, therefore, can be compared on the same scale with the

products obtained from simulations. As expected, as τ_d increases, both the empirical product and the analytical result coincide.

From Eq. (50), it is easy to see that the FPT density function in this limit, up to order 1, is

$$\phi(\tau) = \left[1 + \epsilon \frac{(x_{\text{thr}} - x_0) - \mu\tau}{2D\tau_d} \right] \phi_0(\tau). \quad (51)$$

In fact, instead of working out the limit $\tau_d \rightarrow \infty$ in Eq. (40) as we did before, we can look at the actual physical situation. For $\tau_d \rightarrow \infty$, the exponential drift can be thought of as a constant (i.e., $\tau_d \gg \langle \tau \rangle$). In this case, the constant drift would be $\mu + \epsilon/\tau_d$ and the FPT density function would be given by $\phi_0(\tau)$, Eq. (48), with this modified drift:

$$\phi(\tau) = \frac{(x_{\text{thr}} - x_0)}{\sqrt{4\pi D\tau^3}} \exp \left\{ -\frac{[(x_{\text{thr}} - x_0) - (\mu + \epsilon/\tau_d)\tau]^2}{4D\tau} \right\}. \quad (52)$$

Expanding Eq. (52) around $\epsilon = 0$ up to order 1 gives the same result as obtained before, Eq. (51).

2. Limit $\tau_d \rightarrow 0$

In this limit, the first exponential term between the large curly brackets in Eq. (40), which contains the expression $(s + 1/\tau_d)$, vanishes. The reason is that the real part of the exponent tends to $-\infty$ as τ_d tends to 0. Therefore, the linear correction term of the density function, in the Laplace domain, simplifies to

$$\tilde{\phi}_1^L(s) = -\frac{\mu - \sqrt{\mu^2 + 4Ds}}{2D} \exp \left\{ \frac{(x_{\text{thr}} - x_0)}{2D} \left[\mu - \sqrt{\mu^2 + 4Ds} \right] \right\}. \quad (53)$$

As in the previous limit, the linear term in the expansion of the moments can be obtained directly from Eq. (53) and coincide with the corresponding case in [9].

The simplified expression obtained, Eq. (53), is analytically tractable and the inverse Laplace transform is easy to compute [20]. In the temporal domain, the linear correction is

$$\begin{aligned} \phi_1^{(\tau_d \rightarrow 0)}(\tau) &= \frac{(x_{\text{thr}} - x_0)}{\sqrt{4\pi D\tau^3}} \exp \left\{ -\frac{[(x_{\text{thr}} - x_0) - \mu\tau]^2}{4D\tau} \right\} \\ &\quad \cdot \frac{1}{2D\tau} \left[(x_{\text{thr}} - x_0) - \mu\tau - \frac{2D\tau}{(x_{\text{thr}} - x_0)} \right] \\ &= \frac{\phi_0(\tau)}{2D\tau} \left[(x_{\text{thr}} - x_0) - \mu\tau - \frac{2D\tau}{(x_{\text{thr}} - x_0)} \right]. \end{aligned} \quad (54)$$

In Fig. 5(b) we can observe this limit expression as a function of τ . Additionally, simulations based on different τ_d values show that the limit is reached for sufficiently small values.

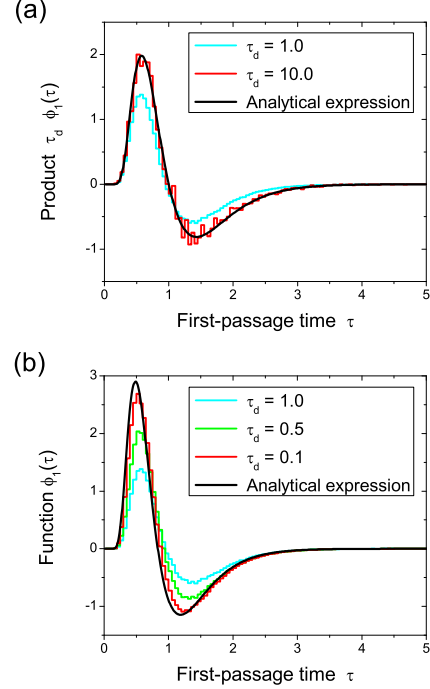


FIG. 5: (Color online) Limit behaviors for the linear correction function $\phi_1(\tau)$. (a) In the limit $\tau_d \rightarrow \infty$ the product $\tau_d \cdot \phi_1(\tau)$ does not depend on τ_d [solid (black) line], Eq. (50). The empirical product is shown for different τ_d values (colored stair-like lines). As τ_d increases, the empirical histogram (obtained as in the insets in Fig. 1 multiplied by the corresponding τ_d) coincides with the analytical expression. (b) In the limit $\tau_d \rightarrow 0$ the density function $\phi_1(\tau)$ does not depend on τ_d [solid (black) line], Eq. (54). The empirical function is shown for different τ_d values (colored stair-like lines). As τ_d decreases, the empirical histogram tends to the analytical expression. Parameters of the simulation as in Fig. 1, with $\epsilon = -0.1$ and different τ_d values.

From Eq. 54, we can compute the FPT density function for this limit which, up to order 1, reads

$$\phi(\tau) = \left\{ 1 + \frac{\epsilon}{2D\tau} \left[(x_{\text{thr}} - x_0) - \mu\tau - \frac{2D\tau}{(x_{\text{thr}} - x_0)} \right] \right\} \phi_0(\tau). \quad (55)$$

The limit $\tau_d \rightarrow 0$ also enables a physical interpretation. Since $\exp[-(t - t_0)/\tau_d]/\tau_d \rightarrow \delta(t_0)$ as $\tau_d \rightarrow 0$, from Eq. (1) it is easy to see that after a differential time from the initial time $t_0 + dt$, the system has moved to the position $x_0 + \epsilon$, and thereafter, the dynamics is governed by a constant drift μ . In this case, the FPT density function is $\phi_0(\tau)$, Eq. (48), with the initial position modified:

$$\phi(\tau) = \frac{(x_{\text{thr}} - x_0 - \epsilon)}{\sqrt{4\pi D\tau^3}} \exp \left\{ -\frac{[(x_{\text{thr}} - x_0 - \epsilon) - \mu\tau]^2}{4D\tau} \right\}. \quad (56)$$

The expansion of Eq. (56) around $\epsilon = 0$ up to order 1

also coincides with Eq. (55).

IV. DISCUSSION AND CONCLUDING REMARKS

In the present work we have analyzed the survival probability and the FPT problem of a Wiener process driven by an exponential time-dependent term superimposed to a constant drift, Eq. (1), in the presence of an absorbing fixed boundary. We first focus on the survival probability in the region of interest and derive the time-inhomogeneous diffusion equation governing it, Eq. (5), in the framework of the backward FP formalism. We propose a solution given by an expansion in terms of the intensity of the exponential drift, Eq. (6), and derive the associated equations and (boundary and initial) conditions to solve each term, Eqs. (8) to (11). Interestingly, the resulting equations are recurrent and easy to solve via a Laplace transformation. We explicitly solve up to the second-order term in the expansion, in the Laplace domain, and give some remarks about higher-order terms (see corresponding subsections). In particular, we show that each term exists, and therefore, the expansion we proposed is justified and constitutes the exact solution. Moreover, when the solution is set to the initial conditions of the problem, the probability depends exclusively on the time elapsed from the initial time, as expected from physical considerations, Eq. (29). The FPT density function is obtained in terms of the survival probability, and we show that the expansion is preserved in this function and its moments, Eqs. (34) and (42). Since the solution of each term is easily obtained in Laplace domain and the inverse transform is not always available, we review some related properties that can be calculated from them: the moments of the density function and the spectral density of an associated renewal process or “spike train”.

In the second part of this work, we focus on the comparison between the explicit results we have obtained and numerical simulations. Since truncation of the series results in an approximate solution, we mainly focus on the first-order expansion. This linear regime coincides with a perturbation scenario. As shown in Fig. 1(a), the first-order term in the expansion of the FPT density completely defines a slightly perturbed system, whereas a higher intensity of the exponential time-dependent drift facilitates higher-order effects [Fig. 1(b)]. Linear expansion of the first two moments of the density function reproduces numerical results accurately, except for low time constants of the exponential drift (Figs. 2 and 3). For a small exponential drift, we calculate the spectral density of the resulting spike train and observe that analytical results coincide extremely well [Fig. 4(b)]. Moreover, the change in the spectral properties due to a small perturbation can be mostly ascribed to the change in

the mean FPT [Fig. 4(c)]. Finally, we derive the behavior of the linear expansion of the FPT density function, in the limit of negligible as well as extremely large time scales, for the time constant of the exponential drift. These limit expressions are inverse Laplace transformed and compared with exact results obtained from physical considerations, valid strictly in the corresponding limit (Fig. 5).

The process considered in this work naturally arises in neuroscience, but it is not restricted to this field (e.g., consider the motion of a charged particle in an exponentially decaying electrical field). In the context of neuroscience, the behavior of stochastic spiking neurons with an adaptation current can be described by stochastic processes with an exponentially decaying temporal term [15]. For spiking neurons, the state variable x corresponds to the membrane potential and its evolution is given by a Langevin equation, where the integration of an input current is performed until a threshold is reached. At this moment a spike is generated and the time corresponds to the FPT in the statistical description [24, 25]. The Wiener process is the stochastic representation of the basic model in theoretical studies, namely, the perfect integrate-and-fire neuron. In addition to external signals, an adapting neuron integrates, between spikes, an exponential time-dependent current corresponding to specific ionic channels [26]. Therefore, the statistical description of the model we have considered provides important measures for analyzing further effects of adaptation on spiking neurons.

Finally, we note some features about the methodology considered here. First, we derive the equation governing the transition probability for a specific temporal drift, Eq. (5), and propose a solution in terms of a certain expansion, Eq. (6). Both procedures can be used for any time-dependent drift. However, in general, the resulting equations would have a source term difficult to tackle analytically. Second, our starting model is the Wiener process with an exponentially decaying temporal drift. It is easy to check that an equivalent formulation can be performed to other one-dimensional processes (e.g., Ornstein-Uhlenbeck process) with the same kind of temporal drifts. In these cases, the only change with respect to the Wiener process is that the homogeneous parts of the differential equations we have obtained, on the left-hand side of Eqs. (8) and (9), are different. Obviously, the difficulty in solving these other cases depends on the system at hand.

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